What is a Weierstrass Point?

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Abstract

On a tropical curve (a metric graph with unbounded edges), one may introduce the so-called "chip-firing game." Given a configuration D of chips on the tropical curve, with possibly negative numbers of chips, one may determine whether it is possible, through a set of approved "moves," f_i to reach a configuration E in which every point on the tropical curve has a nonnegative number of chips. More formally, we may determine which divisors D on the curve are linearly equivalent (via $\sum f_i$) to effective divisors E. We may restrict our attention to starting configurations which have a large number of chips on a single point and some negative chips placed elsewhere in the tropical curve. It turns out that there is a meaningful way to measure how good a given point is at distributing its chips around the curve; points which have a special affinity for this are called *Weierstrass points*. We wish to determine the topological properties of the set of Weierstrass points, namely whether there are finitely many connected components, whether the set of all Weierstrass points is closed, and whether non-smooth Weierstrass points on a bridgeless graph are isolated.

1 The Tropical Semifield

1.1 Definition

We will call $(\mathbb{T}, \oplus, \odot)$ the tropical semifield, where $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ and for $a, b \in \mathbb{T}$,

$$a \oplus b = \max\{a, b\}, \qquad a \odot b = a + b.$$

Some sources also denote tropical operations by placing expressions in quotation marks, writing $a \oplus b =$ "$a + b$" and $a \odot b =$ "ab", though we will not employ this convention (there is also some nuance with equality when graphing equations). When we call $\mathbb{T}$ a *semifield*, we mean that it satisfies all of the field axioms except perhaps the existence of additive inverses and commutativity of multiplication (which we still have). The other axioms are routine to check. Several sources only use the fact that $\mathbb{T}$ is a *semiring*, which additionally drops the condition of existence of multiplicative inverses.

1.2 Dequantization

Some motivation for these tropical operations comes from a process known as *dequantization*, where we essentially look at the image of an object under a logarithm of infinite base, in some sense. To see how our tropical operations arise, let us view how the tropical operations $\oplus$ and $\odot$ arise as limits of logarithms. We know that $(\mathbb{R}_{\geq 0}, +, \cdot)$ is a semi-field, where $\mathbb{R}_{\geq 0}$ consists of all nonnegative real numbers and $+$ and $\cdot$ are the usual addition and multiplication of real numbers. For $t > 1$, the function $\log_t$ is a bijection between $\mathbb{R}_{\geq 0}$ and $\mathbb{T}$, taking $\log(0) = -\infty$. We can thus acquire a semi-field structure $(\mathbb{T}, +_t, \cdot_t)$ by defining

$$x +_t y = \log_t(t^x + t^y), \qquad x \cdot_t y = \log_t(t^x t^y).$$

We already find that $x \cdot_t y = x \odot y$ for any $t > 1$. Since $x \oplus y \leq x + y \leq 2(x \oplus y)$ and $\log_t$ and $x \mapsto t^x$ are increasing, we also have the bound

$$\begin{aligned} x \oplus y &= \log_t(t^{x \oplus y}) \\ &= \log_t(t^x \oplus t^y) \\ &\leq \log_t(t^x + t^y) \\ &= x +_t y \\ &\leq \log_t(2(t^x \oplus t^y)) \\ &= \log_t(2t^{x \oplus y}) \\ &= x \oplus y + \log_t 2 \end{aligned}$$

so as $t \to \infty$, $x +_t y \to x \oplus y$ for any $x, y \in \mathbb{T}$. Hence our tropical operations arise naturally in a limiting process.

To see a geometric realization of this process, we show how we may degenerate a classical line to a tropical one. First, we will briefly look at the process of graphing a tropical line defined by $a \cdot x \oplus b \cdot y \oplus c$. The line will consist of all points such that two terms of the expression are attaining the maximum simultaneously. One may verify that the resulting curve will have the point $(c - a, c - b)$, from which sprout three rays, one west, one south, and one northeast.

Let us consider the line defined by the equation $x - y + 1 = 0$. To work in $\mathbb{R}_{\geq 0}$, we "fold" the line into the first quadrant by taking its image under the map $(x, y) \mapsto (|x|, |y|)$. For $t > 1$, we may then consider the image of the folded line under $(x, y) \mapsto (\log_t x, \log_t y)$. Lastly, we let $t \to \infty$ to find that we have the tropical line defined by $x \oplus y \oplus 1$. A similar situation would occur with other curves defined by similar types of equations, switching between tropical and classical operations.

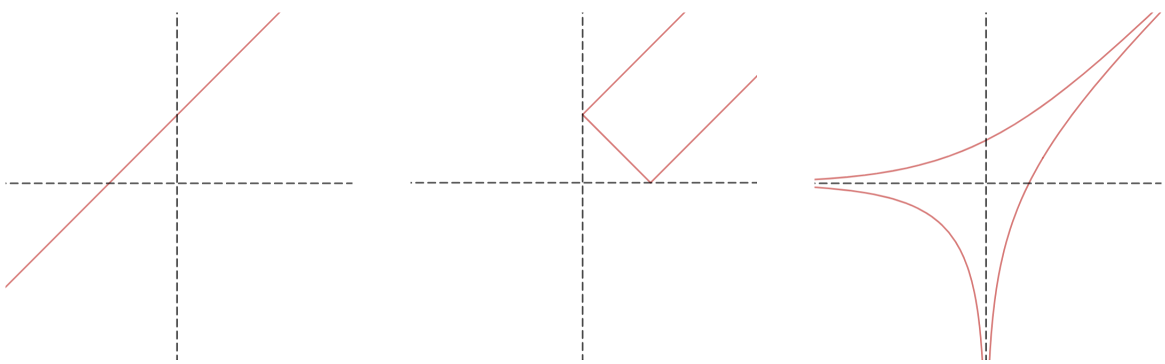

Figure 1: The image of the line under $(x, y) \mapsto (|x|, |y|) \mapsto (\log_t |x|, \log_t |y|)$.

Some call this process "dequantization" to call a comparison with quantum versus classical mechanics. We may view $(\mathbb{R}_{\geq 0}, +, \cdot)$ as a deformation, in a sense, of $(\mathbb{T}, \oplus, \odot)$, which would make $\mathbb{T}$ a dequantization of $\mathbb{R}_{\geq 0}$, if we think of $\mathbb{R}_{\geq 0}$ as quantum mechanics and $\mathbb{T}$ like as classical mechanics.

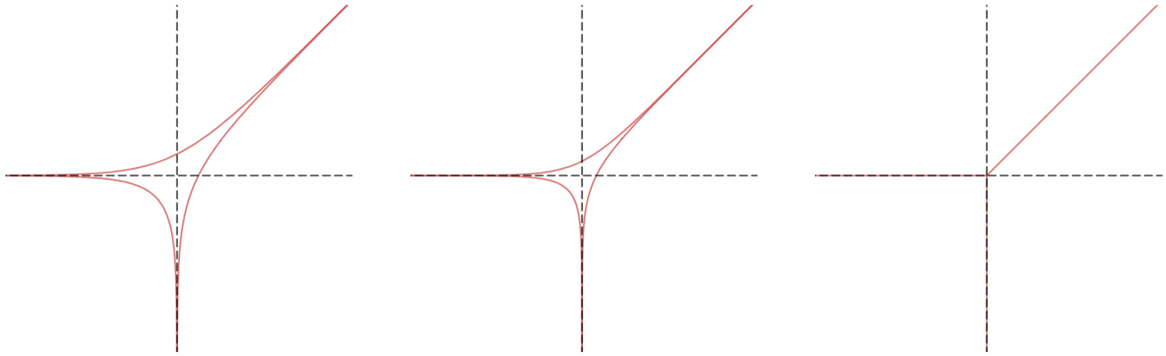

Figure 2: We let $t \to \infty$ to watch the tropical line arise.

2 Divisors and Linear Systems on Graphs

2.1 The Chip-Firing Game

We will be considering situations in which we place a certain (possibly negative) number of "chips" on each vertex of a graph. Our goal is to, through a series of approved "moves," reach a configuration from our starting one such that no vertex of the graph has a negative number of chips; no chip is "in debt." A configuration from which this can be done is called "winnable." The approved moves all have the same form: we designate a vertex of the graph, which will donate one of its chips to each of its neighbors.

This hand-waiving should introduce some of the terminology which may give short names for concepts analogous to more rigorous notions. Chip-firing can also provide intuition, in addition to easing terminology. Because of this, we will endeavor to maintain use of the chip-firing analogy throughout our investigations.

2.2 Formal Definitions

Let $G = (V, E)$ be a finite graph with no loops. The genus g of any graph is defined by $g = |E| - |V| + 1$ and represents the number of independent cycles.

Definition: A *divisor* on G is a member of the free abelian group on V, namely a finite combination of vertices of G. We will call the set of all divisors on a graph by $\mathrm{Div}(G)$. For $v \in V$ and $D \in \mathrm{Div}(G)$, we will call the coefficient of D at v by $D(v)$. A divisor D is *effective* if $D(v) \geq 0$ for all $v \in V$. The set of effective divisors on G will be denoted by $\mathrm{Div}^+(G)$. The *degree* of a divisor D is $\sum_{v \in V} D(v)$. For $k \in \mathbb{Z}$, the set of divisors of degree k will be denoted $\mathrm{Div}_k(G)$.

Building the analogy to our chip-firing game, a divisor corresponds to a chip configuration, and we want to determine whether a given divisor bears some form of equivalence to an effective divisor, a configuration in which there is no debt. The degree is simply the sum of the number of chips on the graph.

We now want to determine what corresponds to a chip-firing move. For a function $f : V \to \mathbb{Z}$, we will define the *Laplacian* $\Delta(f)$ to be the divisor on G defined by

$$\Delta(f) = \sum_{v \in V} \left(\sum_{vw \in E} (f(w) - f(v)) \right) v.$$

Essentially, heights of adjacent vertices are compared, and a number of chips equal to the height difference along an edge slide from the higher end to the lower end. Any divisor arising as a Laplacian of one of these functions is called a *principal divisor*; the set of all such divisors on G is denoted $\mathrm{Prin}(G)$. A single donation move as described earlier would correspond to a function taking the value 1 at the donating vertex and 0 elsewhere. We also note that, for $f, g : V \to \mathbb{Z}$, we have $\Delta(f+g) = \Delta(f) + \Delta(g)$.

Definition: Two divisors $D, D' \in \mathrm{Div}(G)$ are said to be *linearly equivalent*, written $D \sim D'$, if there exists some $f : V \to \mathbb{Z}$ such that $D = D' + \Delta(f)$.

It is not difficult to verify that degree is additive and that all principal divisors have degree 0, so having the same degree is a necessity for linear equivalence, though it is not sufficient in general. This means that there is some interesting theory behind the *Picard group* $\mathrm{Pic}(G) = \mathrm{Div}(G)/\sim$ and *Jacobian group* $\mathrm{Jac}(G) = \mathrm{Div}_0(G)/\sim$. It is not so hard to prove via the matrix-tree theorem that $|\mathrm{Jac}(G)|$ is the number of spanning trees of G. We may verify this fact in a special case.

Proposition 1: All divisors of the same degree on a tree $T = (V, E)$ are linearly equivalent.

Proof. First, we show that all divisors consisting of a single point are linearly equivalent. Let $u, v \in V$. Let $(u, w_n, \ldots, w_1, v)$ be the unique path connecting u and v. Define $f : V \to \mathbb{Z}$ by letting $f(u) = n+1$, $f(v) = 0$, and $f(w_i) = i$ for each $i = 1, \ldots, n$. Let $f(w) = n+1$ for all w in the components of $T - u$ not containing v (or $w_1, \ldots, w_n$). Similarly, let $f(w) = 0$ for all w in components of $T - v$ not containing u. Lastly, for each w in each component of $T - w_i$ not containing u or v, let $f(w) = i$. This will have the effect of $\Delta(f) = v - u$, so $v = u + \Delta(f)$, hence $u \sim v$. Thus two divisors consisting of a single vertex are linearly equivalent.

Now let $D, D' \in \mathrm{Div}(T)$. We may write them out as follows:

$$D = \sum_{i=1}^{m} u_i - \sum_{i=m+1}^{m+n} u_i$$
$$D' = \sum_{i=1}^{m} v_i - \sum_{i=m+1}^{m+n} v_i$$

as they have the same degree, and if D has fewer positive terms than D', then we may pick $u \in V$ and add u term and a $-u$ term to the positive and negative sums until D and D' have the same number of positive and negative terms. For each $i = 1, \ldots, m$, let $f_i : V \to \mathbb{Z}$ such that $u_i = v_i + \Delta(f_i)$. Let $f = \sum_{i=1}^{m} f_i - \sum_{i=m+1}^{m+n} f_i$. Then $D = D' + \Delta(f)$. Hence $D \sim D'$. $\square$

Definition: Let $D \in \mathrm{Div}(G)$. The *linear system associated to D on G*, written $|D|$, is defined

$$|D| = \{E \in \mathrm{Div}^+(G) : D \sim E\}$$

and the *rank* of D, denoted $r(D)$, is

$$r(D) = \begin{cases} -1, & |D| = \varnothing \\ \max\{k : |D - E| \neq \varnothing \text{ for all } E \in \mathrm{Div}_k^+(G)\}, & |D| \neq \varnothing \end{cases}.$$

Some sources may further denote

$$R(D) = \{f : V \to \mathbb{Z} \mid D + \Delta(f) \in \mathrm{Div}^+(G)\}.$$

2.3 Example of Winning a Configuration

Our graph G is given with starting configuration D, having its coefficients listed by the associated vertices. I claim that this configuration is winnable; I buttress my claim by exhibiting a function f whose value on each vertex is shown in the diagram. In terms of chip firing, I started with the zero function, then increased the value by 1 at a vertex each time it donates to its neighbors and decreased the value by one each time it borrows. I then calculate the Laplacian $\Delta(f)$ by adding up the differences between the value of f at a vertex and the value of f at the ends of each of its edges. This calculates how many chips each vertex sent or received. For example, the left-most vertex sends and receives a chip from the right-most (as both have value 1), loses one chip to the bottom left vertex ($0 - 1 = -1$), and loses two chips to the top vertex ($-1 - 1 = -2$). Hence $\Delta(f)$ has coefficient -3 at the far left, meaning we lose three chips there. Lastly, we calculate $D + \Delta(f)$ to determine our new chip configuration after firing and find that the result is effective, meaning we have won the game.

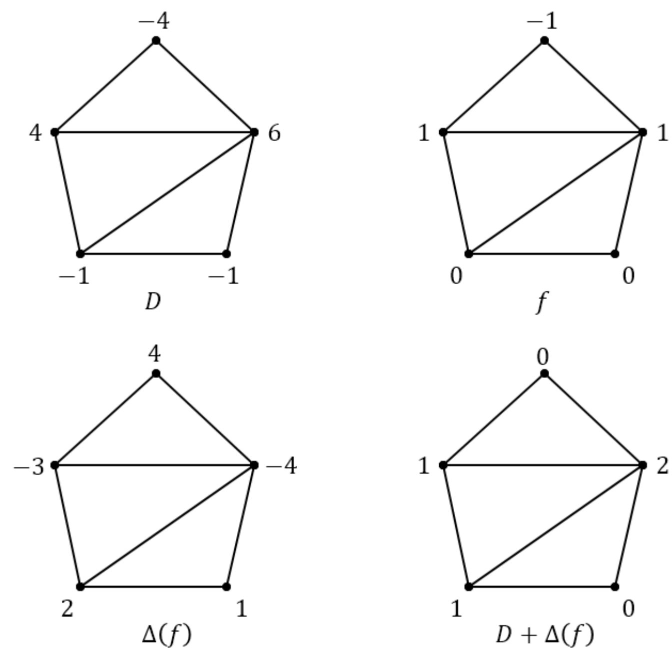

Figure 3: Our starting configuration, chip-firing move, resulting change in chips, and final effective configuration, showing that the configuration is winnable.

3 Divisors and Linear Systems on Tropical Curves

3.1 Definitions and Analogies

We now want to develop a chip-firing game like the one we had on finite graphs, but now in a continuous setting.

Definition: A *tropical curve* is a connected metric graph with possibly unbounded edges. We will allow only for finitely many *non-smooth* points, namely points of valence greater than 2, and only finitely many edges. All unbounded edges will have points at their ends to make our tropical curve compact.

Let Γ be a tropical curve. As before, divisors, members of $\text{Div}(\Gamma)$, will be elements of the free abelian group on Γ. The notions of degree and effective divisors will carry over as well. We must now determine how we are allowed to fire chips on Γ.

Definition: A *tropical rational function* is a continuous function $f : \Gamma \to \mathbb{R}$ which has finitely many linear pieces of integer slope. For $x \in \Gamma$, define $\text{ord}_f(x)$, the *order of f at x*, to be the sum of the outgoing slopes of f from x. When $\text{ord}_f(x) > 0$, we call x a *zero* of f, and when $\text{ord}_f(x) < 0$, we call x a *pole* of f. The *Laplacian* $\Delta(f)$ is the divisor

$$\Delta(f) = \sum_{x \in \Gamma} (\text{ord}_f(x))x.$$

Having established this reworking of chip-firing, our definitions for linear equivalence, linear systems, and rank on a graph now extend to tropical curves as well. We notice that $\Delta(f \odot g) = \Delta(f) + \Delta(g)$, similarly to before.

The set $R(D)$ for $D \in \text{Div}(\Gamma)$ is much like it was on graphs, with tropical rational functions in this case. What we find is that $R(D)$ actually has a semi-module structure over $\mathbb{T}$, taking $\oplus$ as addition and $\odot$ pointwise as scalar multiplication.

3.2 The Circle

Let us do a brief investigation of linear equivalence on the simplest tropical curve which isn't a tree: the circle. We may first notice that no two distinct divisors consisting of a single point are linearly equivalent. If you fire a chip a distance t from x to y on the circle via a tropical rational function f, then $f(y) = f(x) - t$; there can be no disturbances along the path, as we can afford to deposit no extra chips or take any from the arc between x and y. However, since we've used up our only chip, f must be constant along our other path from x to y, putting a jump discontinuity at y or somewhere along this second path.

3.3 Riemann Surfaces

A *Riemann surface* is a connected one-dimensional complex manifold. Like with tropical curves, the surfaces studied are generally compact. Linear systems on Riemann surfaces have served as the initial point of the theory, as many nice properties of Riemann surfaces have analogs in other settings, a good example being the *Riemann-Roch Theorem*, which will be central to the study of Weierstrass points. Analogously to tropical rational functions, we deal with meromorphic functions on the surface, with the terms "zero" and "pole" from earlier being suggestive. Instead of thinking of the rank of a divisor D and the semi-module $R(D)$, we look at the dimension of the vector space $L(D)$ of all meromorphic functions f such that $D + \Delta(f)$ is effective.

4 Weierstrass Points

4.1 The Riemann-Roch Theorem

Definition: Let $G = (V, E)$ be a graph and Γ be a tropical curve. The *canonical divisor* on G is $\sum_{v \in V}(\text{val}(v) - 2)v$ and on Γ is $\sum_{x \in \Gamma}(\text{val}(x) - 2)x$. The canonical divisor on a Riemann surface is picked from the linear equivalence class of all divisors of global meromorphic 1-forms.

Theorem: (*Riemann-Roch*) Let Γ be a graph, tropical curve, or Riemann surface of genus g. Let D be a divisor on this object and let K be the canonical divisor. Then

$$r(D) - r(K - D) = \deg(D) - g + 1$$

where $r(D)$ is the rank of D on a graph or tropical curve and the dimension of $L(D)$ on a Riemann surface.

This result was first stated by Bernhard Riemann in 1857 as Riemann's Inequality, stating $r(D) \geq \deg(D) - g + 1$ for D a divisor on a Riemann surface. Gustav Roch refined it 1865 to become an equality with an error term.

4.2 Rank and Gap Sequences

Since degree is invariant under linear equivalence and all effective divisors have nonnegative degree, we know that $r(D) = -1$ for a divisor D of negative degree on a graph or tropical curve. We may also observe that the canonical divisor, on a graph $G = (V, E)$ with genus g for example, has degree

$$\deg(K) = \sum_{v \in V}(\text{val}(v) - 2) = \sum_{v \in V} \text{val}(v) - \sum_{v \in V} 2 = 2|E| - 2|V| = 2(|E| - |V| + 1) - 2 = 2g - 2$$

and similarly for tropical curves (this number is the same on Riemann surfaces, though the argument is not so clearly similar). Hence, for a divisor D with $\deg(D) \geq 2g - 1$, we have $\deg(K - D) \leq -1$, so $r(K - D) = -1$, meaning that $r(D) = \deg(D) - g$.

Definition: Let Γ be a tropical curve. Let $P \in \Gamma$. The *rank sequence* at P is $(r(nP))_{n \in \mathbb{Z}_{\geq 0}}$.

A further observation we can make is that increasing n by one increases $r(nP)$ by either one or zero. That the rank sequence is nondecreasing is clear by elementary considerations; if we have $E \in \text{Div}^+(\Gamma)$, then any chip-firing move which shows that $nP - E$ is winnable works just as well for $(n + 1)P - E$. That $r(nP)$ increases by only one at most comes from the fact that $r(K - nP)$ is non-increasing and $r(nP) = n - g + 1 - r(K - nP)$ has only the n term increase by 1.

This allows us to fairly well characterize any rank sequence: it always increases by 1 one n exceeds $2g - 1$ and increases on half of $n = 1, 2, \ldots, 2g - 2$. A natural question is to ask where the increases in rank occur for given points on a tropical curve. We will ask the complementary question by trying to pinpoint the *gap sequence* for $P \in \Gamma$, namely the $n \in \{1, \ldots, 2g - 1\}$ such that $r(nP) = r((n - 1)P)$.

Definition: A *non-Weierstrass point* on a graph, tropical curve, or Riemann surface is a point whose gap sequence is $1, 2, \ldots, g$. A point which has any other gap sequence is called a *Weierstrass point*. The *weight* of a gap sequence $a_1, \ldots, a_g$ is $\sum_{i=1}^{g}(a_g - g)$.

If we wish to determine whether a point P is a Weierstrass point or not, we must simply calculate $r(gP)$; it will be positive if and only if P is a Weierstrass point. There is only interesting theory of Weierstrass points if $g \geq 2$, as we cannot have Weierstrass points otherwise. It is established for Riemann surfaces that the sum of the weights of all points on the surface is $g(g^2 - 1)$. This shows two nice features; the number of Weierstrass points is always finite and positive for surfaces of genus at least 2.

4.3 Example Calculation

We are going to try to verify that the point P on the given curve Γ is a Weierstrass point by determining that $r(3P) > 0$. This means that, given 3 chips on P and a point x on Γ, we should be able to come up with a chip-firing move which creates no debt and lands at least one chip on x. This will verify that $|3P - E| \neq \varnothing$ for all $E \in \mathrm{Div}_1^+(\Gamma)$, since effective divisors of degree one consist of only one point.

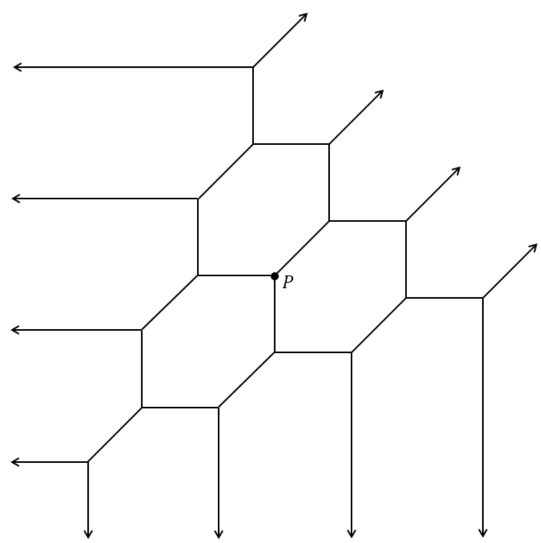

Figure 4: This curve looks like a the locus for a degree 4 tropical polynomial.

Assume that all edges are of equal lengths. Moreover, since all of the points on the unbounded edges are linearly equivalent, we may ignore the unbounded edges and focus on the points at their finite ends. The unbounded edges were included only to increase the resemblance of Γ to a tropical plane curve arising from a tropical polynomial in two variables.

Once we have cut off the "tree-looking" portions of the curve, what we have left can be deformed into a more symmetrical structure via planar isometry. Although a circle is slightly misleading, it will serve out purposes nicely; the outside edges should actually bulge out to truly be 4 times as long as the radii, but this will cause no issue.

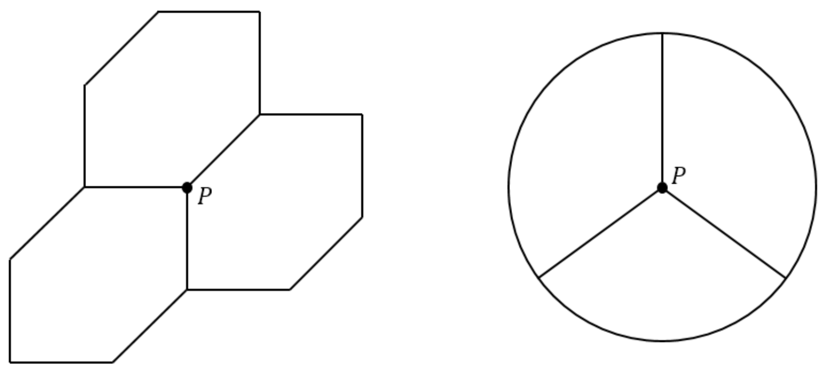

Figure 5: Our curve Γ without unbounded edges and its deformation to a more symmetrical graph.

This redrawing of the curve will make it easier to break our chip-firing games into two types: those with x on a radial edge and those with x along the outer ring. We give diagrams showing the chip-firing moves for both of these.

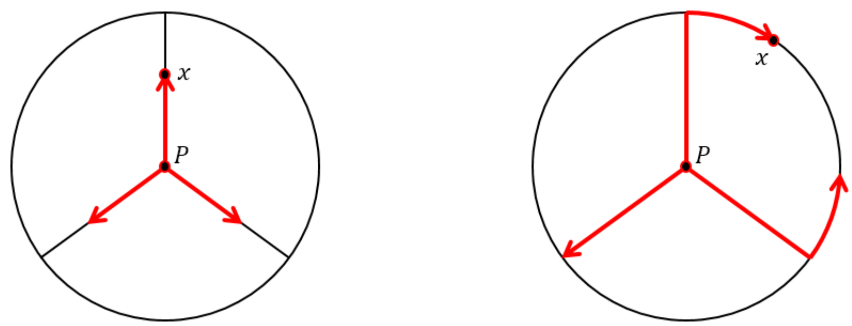

Figure 6: The chip-firing moves which win the different games we could face.

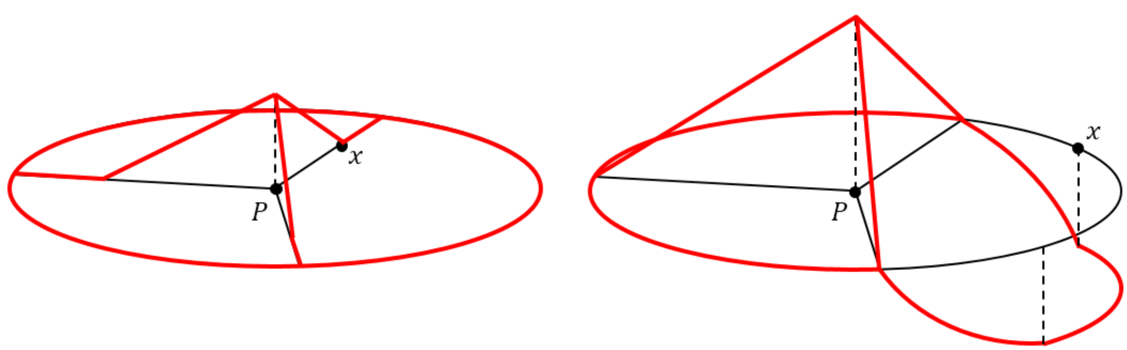

Figure 7: An oblique view of our chip-firing moves; the height of the red line above the graph is the value taken by the tropical rational function giving rise to the chip-firing move.

To interpret the diagrams, consider the tropical rational function giving rise to the chip-firing to have downward slope 1 along the direction indicated by the arrow. It is constant everywhere else. Hence we want no more than 3 arrows coming out of P and at least one terminating at our point x where we wish to deposit a chip.

To ensure that we have a valid chip-firing move, we verify that the change in the value of the function between any pair of points is independent of the path takes; this checks for jump discontinuities.

The radial case is easy; fire a chip along each radial edge out to the distance between P and x. For the other case, fire a chip all the way along each radial edge. From the two radial edges cutting off the arc containing x, fire the chip further in from each side until one of them hits x.

4.4 The Banana

To see that Weierstrass points can come in intervals, we look to the banana graph B_g of genus g, which, for our purposes, will consist of two non-smooth points joined by $g+1$ unit-length edges. We will claim that the the portion of every edge excluding an open length $1/g$ segment on each end (and the non-smooth points) is entirely Weierstrass points.

Let P be in this described region. We want to show that $r(gP) \geq 1$. Let $x \in B_g$. If x is a non-smooth point or on the same edge as P, we simply fire a chip to x from P, then keep firing the other chips on P to the other non-smooth point until we have "evened out" our tropical rational function. If x is on a different edge than P, we may fire one chip from P to each non-smooth point, then even out our tropical rational function by firing as many chips from P as we need to the closer non-smooth point. Once we have a chip at each non-smooth point, we fire those chips inward at the same speed along the edge with x until one of the chips hits x.

5 Reduced Divisors

5.1 Definition

We may now introduce another tool to help detect, and even better, detect the absence of, Weierstrass points. We will continue calling our tropical curve by Γ. First, we must define the *out-degree* from a subset of Γ at a point. Let X be a reasonable subset of Γ. Let $x \in X$. We define $\deg_X^{out}(x)$ to be the number of disjoint intervals with x as an endpoint which otherwise do not intersect X.

Definition: Let $P \in \Gamma$. A divisor $D \in \mathrm{Div}(\Gamma)$ is said to be P-*reduced* if:

1. For all $x \in \Gamma$ with $x \neq P$, we have $D(x) \geq 0$
2. For all closed subsets X of Γ, we have $D(x) < \deg_X^{out}(x)$ for some $x \in \partial X$

Proposition 2: Given $P \in \Gamma$ and $D \in \mathrm{Div}(\Gamma)$, there is a unique P-reduced divisor linearly equivalent to D, often called D_P.

Because of this fact, given a divisor D, we may construct a function on Γ which sends P to the P-reduced divisor linearly equivalent to D; this map has some nice cell-complex structure, but we won't investigate that here. You can turn to [1] for this characterization.

What reduced divisor can do for us is offer an alternate characterization of Weierstrass points. We recall previous formulations and add some new ones here:

Proposition 3: A Weierstrass point P of Γ (which has genus g and canonical divisor K) is one which satisfies any of the following equivalent conditions:

1. The gap sequence of P is not $1, 2, \ldots, g$
2. $r(gP) \geq 1$

3. $K_P(P) \geq g$

Proof. We only need to handle the equivalence of (2) and (3), since (1) and (2) were already investigated. We will prove the more general formula, that

$$r(gP) + g - 1 = K_P(P).$$

Let $n = K_P(P)$ for ease of writing. Since rank is preserved under linear equivalence, by Riemann-Roch, we have

$$r(nP) - r(K_P - nP) = r(nP) = n - g + 1$$

since $r(K_P - nP) = 0$. This is because $K_P - mP$ is P-reduced for any m, so $K_P - nP$ is P-reduced. This means that $K_P - nP - P$, also P-reduced, is not linearly equivalent to an effective divisor, since we cannot add more chips to P. Hence $K_P - nP$ does not have positive rank because there are points of Γ not in the support of $K_P - nP$. The equivalence of (2) and (3) follows. □

5.2 More Banana

Let A and B be the non-smooth points on B_g. Let P lie on an edge a distance at least $1/g$ away from A and B. Let $K = (g-1)A + (g-1)B$ be the canonical divisor on B_g. Suppose WoLoG that P is not farther from A than it is from B. We may fire all of our $g - 1$ chips from A and a single chip from B to P. To even things out, we fire all of our chips from B along the edge with P as far as we need; this will work, since sending them all to P will be an over-correction, so there is a point in the middle where we have a valid chip-firing.

Suppose now that P is strictly within $1/g$ of A. Fire all chips $g - 1$ from A to P and fire one chip from B along the edge with P as far as needed, to a point x, to even things out; this will result in x being closer to B than P is, since x is less $(g-1)/g$ away from B. Call the resulting divisor D; we claim that D is P-reduced. To see this, we need only look at the subset of B_g whose boundary points are B and x and which does not contain P, since the support of D has only these three points. The resulting set is an interval $[B, x]$ along the edge with P. This set cannot fire, as $\deg^{out}_{[B,x]}(B) = g - 1$ but $D(B) = g - 2$. Since D is effective, it is also P-reduced. Since $D(P) = K_P(P) = g - 1 < g$, P is not a Weierstrass point. Note that this argument works for that non-smooth points, as K is A- and B-reduced.

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